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$$E(0) = \omega(0) - \omega(1) + \cdots + (-1)^k \omega(k)$$

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Solution. If n is a composite number such that, $1 < n < 50$, it must be multiple of 2, 3, 5, or 7. (If p, q are primes so that $pq|n$, and $p, q > 7$, then $pq > 50$.)

A_2 : multiples of 2
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To compute: E_0

$A_2 \cap A_3$: multiples of 6
$\dots$

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Definition (Euler ϕ-function)**.** For $n \in \mathbb{N}$, $\phi(n) := |\{m \in \mathbb{N} \mid m < n, (m, n) = 1\}|$

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Proof. A_i: multiples of p_i, which are less than n
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Proof. A_i: multiples of p_i, which are less than n
$E_0 = \phi(n)$.
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$$\begin{aligned} E(0) &= \omega(0) - \omega(1) + \cdots + (-1)^k \omega(k) \\ &= n - \Sigma\frac{n}{p_i} + \cdots + (-1)^k \frac{n}{p_i p_2 \ldots p_k} \\ &= n\left(1 - \Sigma\frac{1}{p_i} + \cdots + (-1)^k \frac{1}{p_i p_2 \ldots p_k}\right) \\ &= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) \end{aligned}$$

□

Example. $D(n, r, k)$: The number of r permutations of $\{1, 2, \ldots, n\}$ with exactly k numbers fixed. Derive a formula for $D(n, r, k)$

Solution. S : set of r permutations
A_i: Number of r permutations with ith number fixed

Have to find, $E(k)$

$\omega(i_1, i_2, \ldots, i_t) =$

Definition (Euler ϕ-function)**.** For $n \in \mathbb{N}$, $\phi(n) := |\{m \in \mathbb{N} \mid m < n, (m, n) = 1\}|$

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Solution. S : set of r permutations
A_i: Number of r permutations with ith number fixed

Have to find, $E(k)$

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A_i: Number of r permutations with ith number fixed

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□

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Solution. S : set of r permutations
A_i: Number of r permutations with ith number fixed

Have to find, $E(k)$

$\omega(i_1,i_2,\ldots,i_t) = P^{n-t}_{r-t}$
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Solution. S : set of r permutations
A_i: Number of r permutations with ith number fixed

Have to find, $E(k)$

$\omega(i_1,i_2,\ldots,i_t) = P^{n-t}_{r-t}$
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$$
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E(k) &= \omega(k) - \binom{k+1}{1}\omega(k) + \binom{k+2}{2}\omega(k+1) \\
&\quad - + \cdots + (-1)^{r-k}\binom{r}{k}\omega(r) \\
&=
\end{aligned}
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E(k) &= \omega(k) - \binom{k+1}{1}\omega(k) + \binom{k+2}{2}\omega(k+1) \\
&\quad - + \cdots + (-1)^{r-k}\binom{r}{k}\omega(r) \\
&= \binom{k}{0}\binom{n}{k}P^{n-k}_{r-k} - \binom{k+1}{1}\binom{n}{1}P^{n-k-1}_{r-k-1} + \binom{k+2}{2}\binom{n}{2}P^{n-k-2}_{r-k-2} \\
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\end{aligned}
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Exercise. $\lim_{n\to\infty} D(n, n, 0)/n! = e^{-1}$

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