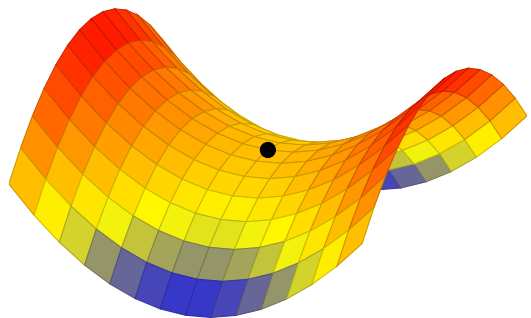
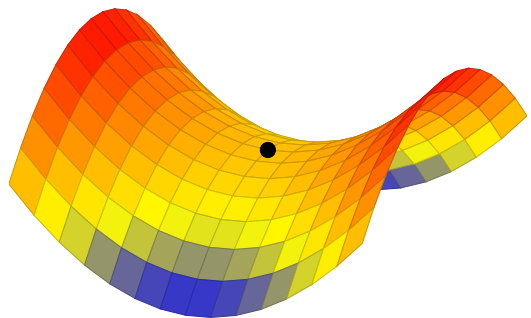


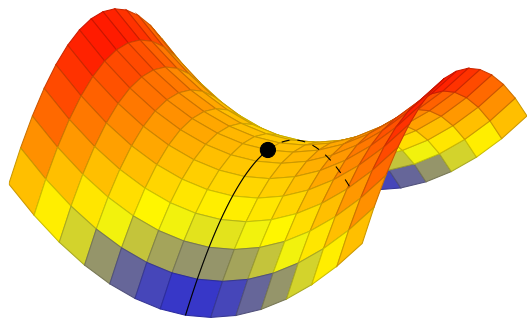
Consider this saddle shaped surface



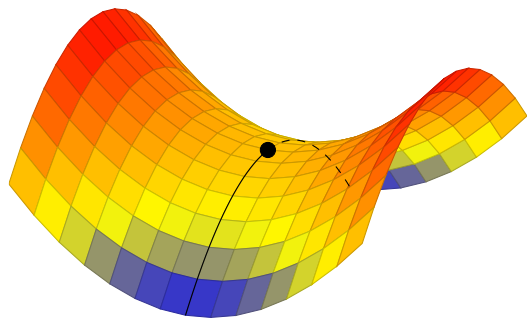
and this point on it



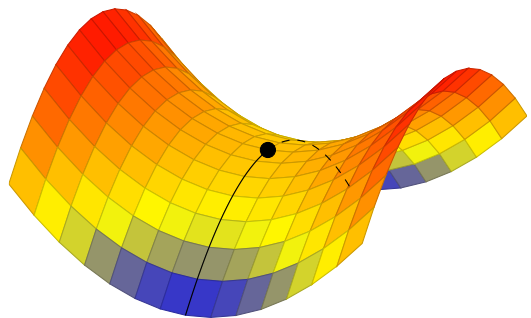
If we assume that the normal points in the upward direction



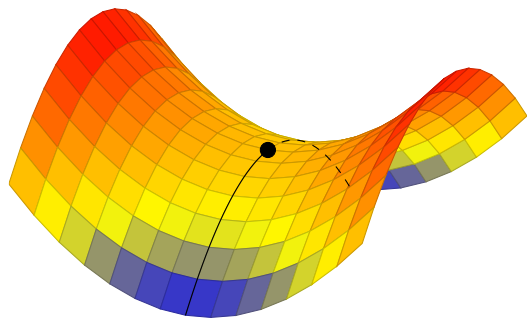
then this curve has negative normal curvature on that point



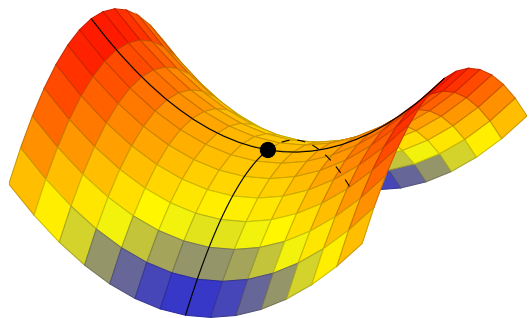
In fact, it is the smallest possible normal curvature at this point



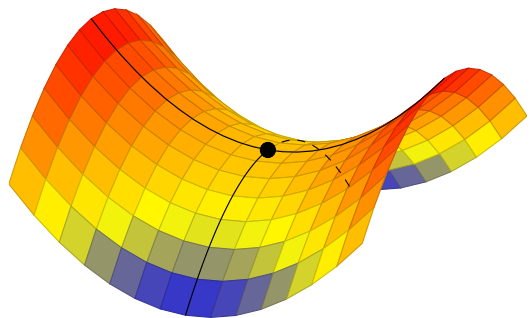
and, therefore, is one of the principal curvatures.



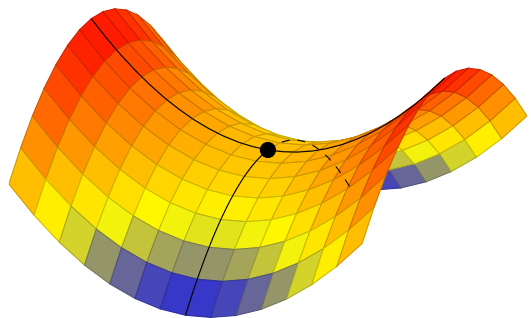
and its velocity vector is the principal direction



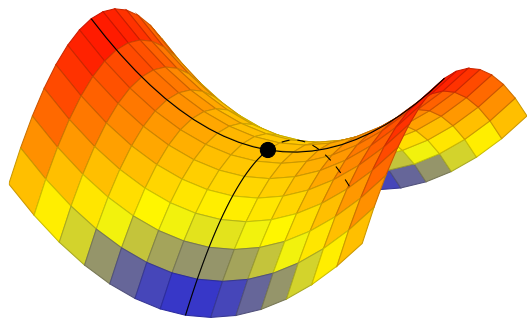
Now consider this curve. It has positive curvature at that point



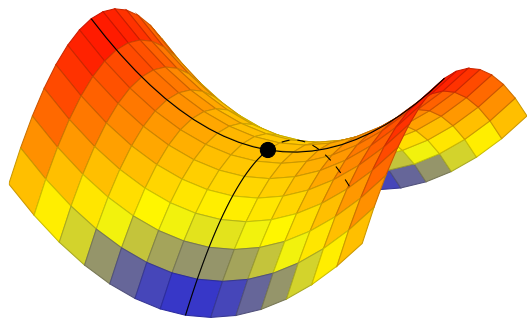
and is the maximum possible normal curvature at that point



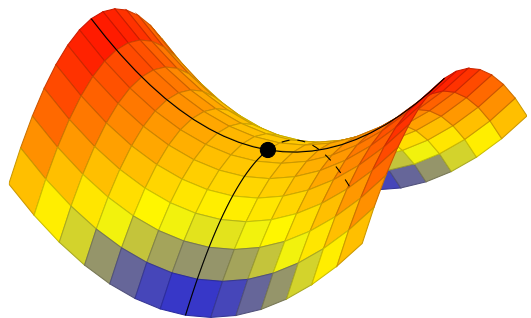
and, therefore, is the other principal curvature.



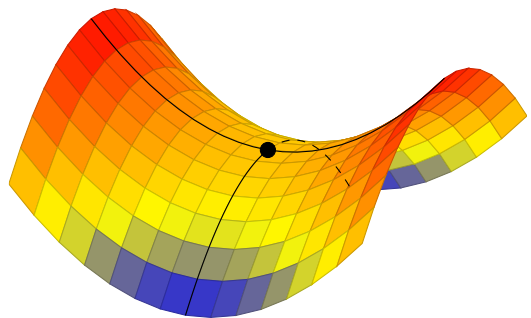
and its velocity vector is the other principal direction



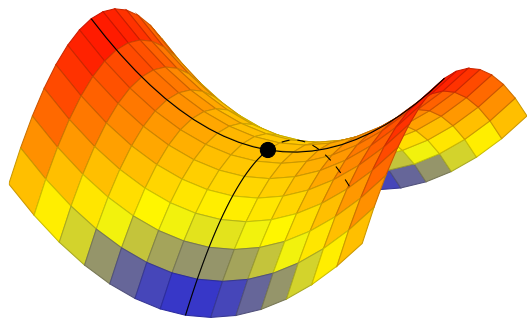
Note that both directions are perpendicular to each other



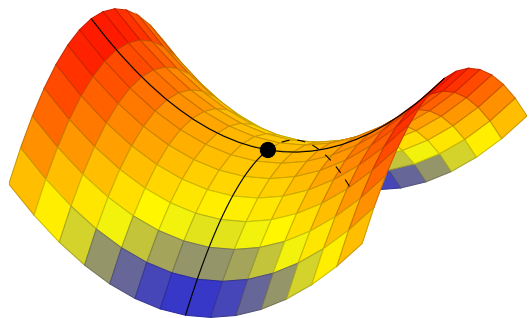
and that the Gaussian curvature is negative because the principal curvatures have different signs



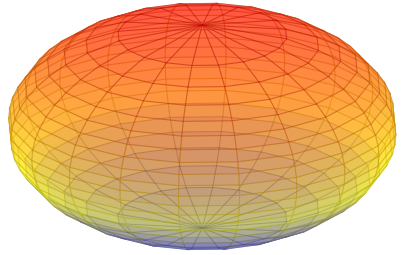
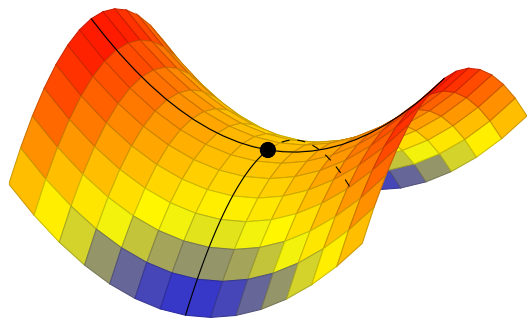
On the other hand, at the north pole of the sphere, all curves have the same normal curvature



no matter which direction they point in.



So its principal curvatures are the same



and whether they are both positive or both negative, the product will be positive

$f : S_1 \rightarrow S_2$ smooth

To fully appreciate the implication of Gauss' theorem Egregium

$f : S_1 \rightarrow S_2$ smooth

let us study functions between surfaces.

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

and how they change the arc-length of curves

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

i.e. the arc-length of the curve on S_2 .

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt$$

Recall the formula for arc length

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{d}{dt}f(\gamma(t)) \cdot \frac{d}{dt}f(\gamma(t))} dt$$

Recall that $\frac{d}{dt}f(\gamma(t))$ at t

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt$$

depends only on the velocity of γ at t

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \end{aligned}$$

and on what is denoted as $D_p(f)$

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt\end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

We now define what is called the “pull back” of the first fundamental form

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt \end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

and see that the arc-length can be written in terms of it

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt\end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

As before, this is telling us that if we find compute $f^* \langle \cdot, \cdot \rangle_p$ point on the surface

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt\end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

We only need to know the velocity vector field of any curve whose arc-length we wish to compute

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

arc-length of $f \circ \gamma =$ arc-length of γ

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt\end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

We now have an alternative criterion for when a function preserves the arc length of a curve

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

arc-length of $f \circ \gamma =$ arc-length of γ

if $\int_{t_0}^{t_1} f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt \end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

By definition it is when these integrals are the same

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

arc-length of $f \circ \gamma =$ arc-length of γ

if $\int_{t_0}^{t_1} f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$

if $f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_p$ for all $p \in S_1$.

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt \end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

And that will happen when these two forms are the same

$f : S_1 \rightarrow S_2$ smooth

$\gamma : (\alpha, \beta) \rightarrow S_1$

$f \circ \gamma : (\alpha, \beta) \rightarrow S_2$

arc-length of $f \circ \gamma =$ arc-length of γ for *any* γ
if **and only if** $\int_{t_0}^{t_1} f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle dt$
if **and only if** $f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_p$ for all $p \in S_1$.

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{d}{dt} f(\gamma(t)) \cdot \frac{d}{dt} f(\gamma(t))} dt \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) \cdot D_{\gamma(t)}(f) \dot{\gamma}(t)} dt \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} dt \end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} \cdot D_p(f) \mathbf{w} \rangle_{f(p)}$$

The converse is true if and only if f preserves arc-length of *any* curve