Hints / Solutions to Exercise sheet 4

Curves and Surfaces, MTH201

Question 1: Prove that $\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)) = -\dot{\gamma}(t).\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$. Here $\hat{\ltimes}$ denotes the normal to a surface. (*Hint*: You have done similar things many times before. Which rule helps you?)

Solution 1: Note that $\dot{\gamma}(t)$ is tangent to the surface because because it is the velocity vector of a curve, namely γ, which lies on the surface. So it is perpendicular to the unit normal to the surface. Therefore, its dot product with it is 0, i.e.

$$\dot{\gamma}(t).\hat{n}(\gamma(t)) = 0$$

Differentiating, and using the product rule,

$$\ddot{\gamma}(t).\hat{n}(\gamma(t)) + \frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t).\hat{n}(\gamma(t)) = 0$$

Question 4: Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$, where $\mathbf{T}(t)$ is the unit tangent of γ at t.

Solution 4: The unit normal is defined as the cross product of σ_x and σ_y and by the property of cross products, is perpendicular to each. Chain rule shows that any tangent vector is a linear combination of σ_x and σ_y, and so every tangent vector is also perpendicular to the unit normal.

Question 6: Consider a parametrization, $\gamma(t)$ of a curve on a surface and let $\mathbf{N}(t)$ denote its unit normal at t. Prove that $\mathbf{N}(t) = \pm\hat{\mathbf{n}}(\gamma(t))$ if and only if $\kappa_g(t) = 0$

Solution 4: The unit normal of the curve is in the direction of the acceleration of a unit speed parametrization. So, the unit normal to the curve is in the direction of the unit normal of the surface if and only if the acceleration is in the direction of the unit normal. Also, the acceleration is in the direction of the unit normal means it is equal to the component in the direction of the normal and has no other component, which is equivalent to the geodesic curvature being 0.

Question 6: Consider the sphere of radius 1. Give it a surface patch and use that to compute the normal curvature of a unit speed parametrization of any curve on the sphere at any given point.

Solution 6: Owing to the symmetry of the sphere, it deos not matter which point we choose. Also, recall that the normal curvature of any two unit speed parametrizations depend only on the direction of its (unit) velocity vector. Furthermore, note that on a sphere, owing to its symmetry, the direction too does not matter. Therefore, we can choose any curve on the sphere and find its component in the normal direction.

First let us compute the normal,

$\sigma(x,y) = (x, y, \sqrt{1-x^2-y^2})$ defines a surface patch.

$$\sigma_x(x,y) = (1, 0, \frac{x}{\sqrt{1-x^2-y^2}})$$

$$\sigma_y(x,y) = (0, 1, \frac{y}{\sqrt{1-x^2-y^2}})$$

$$\sigma_x(x,y) \times \sigma_y(x,y) = (\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}, 1)$$

$$\hat{n}(x,y) = (-x, -y, \sqrt{1-x^2-y^2})$$

Note how this vector is normal to the position vector defining a point on the part of the surface covered by the surface patch.

We have already noted that it does not matter which unit speed parametrization we choose. Therefore, we may choose. In fact, all curves on the surface are of the form

$$\gamma(t) = \sigma(x(t), y(t)) = (x(t), y(t), \sqrt{1-x^2(t)-y^2(t)})$$

for some $x(t)$ and $y(t)$. Let us choose $x(t) = \cos(t)$ and $y(t) = \sin(t)$ so that the parametrization is,

$$\gamma(t) = \sigma(x(t), y(t)) = (\cos(t), \sin(t), 0)$$

Note that this is just the great circle lying on the sphere and the paramtrization is unit speed

Now,

$$\ddot{\gamma}(t) = (-\cos(t), -\sin(t), 0)$$

Consider the point at $t = 0$, i.e. $\gamma(0) = (1, 0, 0)$. Note that the unit normal at that point is $(1, 0, 0)$ and $\ddot{\gamma}(0) = (-1, -1, 0)$. Taking the dot product, we get that the normal curvature is -1.