Hints / Solutions to Exercise sheet 8

Curves and Surfaces, MTH201

Question 2: Prove that the geodesic curvature of a curve in a plane (treated as a surface in $\mathbb{R}^3$) is equal to the plane curvature.
Solution 2:
The main idea is to show that the acceleration vector will point in the tangent plane, therefore, the normal component of the acceleration is 0 (that is the component used to compute the normal curvature). The other component, whose magnitude is the geodesic curvarure, is the same as the acceleration itself, whose magnitude is the usual curvature.

Consider a plane $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$. We can give it a surface patch by $\sigma(x,y)=(x,y,-\frac{a(x-x_0)+b(y-y_0)}{c}+z_0)$. The idea behind this surface patch is to solve for z, in terms of x and y, which is only possible if $c\neq 0$ but if not, one can solve for y or x instead).

$$\sigma_x = (1,0,-\frac{a}{c})$$
$$\sigma_y = (0,1,-\frac{b}{c})$$

Note that $\sigma_x \times \sigma_y = (a/c, b/c, 1)$ is parallel to (a,b,c).

Any curve is of the form,
$\gamma(t)=\sigma(x(t),y(t))=(x(t),y(t),-\frac{a(x(t)-x_0)+b(y(t)-y_0)}{c}+z_0)$
$\dot{\gamma}(t)=(x'(t),y'(t),-\frac{ax'(t)+by'(t)}{c})$
Note, $(a,b,c).\dot{\gamma}(t)=0$

The crucial observation is that,
$\ddot{\gamma}(t)=(x''(t),y''(t),-\frac{ax''(t)+by''(t)}{c})$
and so, $(a/c,b/c,1).\ddot{\gamma}(t)=0$. The acceleration is also perpendicular to $\sigma_x\times\sigma_y$, so is perpendicular to the unit normal. Therefore, the normal component is 0, and the geodesic component is the entire acceleration vector.
Question 3: Compute the normal curvature of any curve on the sphere at a point in the region covered by the surface patch, $\sigma(x,y)=(x,y,\sqrt{1-x^2-y^2})$. Can you interpret the answer physically? Using this, prove that curves on the sphere that have constant geodesic curvature are circles.
Solution 3:
This may be solved without using the surface patch.

We know that any curve with the same velocity vector at p will have the same normal curvature at p. We can find a great circle (i.e. a circle obtained by intersecting the sphere with a plane passing through the center) passing through p with velocity vector in any given direction. The acceleration vector of a circle points toward the center and has magniture 1/radius. Since this is a great circle, its center is the center of the sphere and radius is the radius of the sphere which is 1. So the normal curvature at any point and for any curve given by a unit speed parametrization is 1.

Since $\kappa^2 = \kappa_n^2 + \kappa_g^2$, if the curve also has constant geodesic curvature, it will also have constant curvature.

To prove that it is part of a circle, would be then equivalent to showing that it lies on a plane. That in turn, is equivalent to showing that the torsion is 0. Torsion first appeared in Frenet-Serret, so let us keep that as a guide.

Until now, we know two facts about the curve: it lies on the sphere, and it has constant curvature.

The fact that curve lies on the sphere tells us, $\|\gamma(t)\|^2 = 1$
$\mathrm{d}dt(\gamma(t).\gamma(t)) = 0$
$2\mathbf{T}(t).\gamma(t) = 0$

If we differentiate it again, it will involve the curvature and we can incorporate the fact that the curvature is constant: $\dot{\mathbf{T}}(t).\gamma(t) + \mathbf{T}(t).\mathbf{T}(t) = 0$
$cN(t).\gamma(t) + 1 = 0$

If we differentiate again, we will get some relation involving the torsion.
$(-c\mathbf{T} + \tau(t)\mathbf{B}(t)).\gamma(t) + \mathbf{N}(t).\mathbf{T}(t) = 0$
$-c\mathbf{T}.\gamma(t) + \tau(t)\mathbf{B}(t).\gamma(t) + \mathbf{N}(t).\mathbf{T}(t) = 0$
$-c0 + \tau(t)\mathbf{B}(t).\gamma(t) + 0 = 0$
$\tau(t)\mathbf{B}(t).\gamma(t) = 0$

Now if $\mathbf{B}(t).\gamma(t) \neq 0$ for any t, then the $\tau(t) = 0$ and we are done. If not, differentiating both sides of $\mathbf{B}(t).\gamma(t) = 0$ and using Frenet-Serret will impose a relationship between $\mathbf{T}$, $\mathbf{N}$, and γ:
$\mathbf{B}(t).\gamma(t) = 0$
$\dot{\mathbf{B}}(t).\gamma(t) + \mathbf{B}(t).\mathbf{T}(t) = 0$
$-\tau(t)\mathbf{N}(t).\gamma(t) + \mathbf{B}(t).\mathbf{T}(t) = 0$
$-\tau(t)\mathbf{N}(t).\gamma(t) = 0$

Once again, either $\tau(t) = 0$ or $\mathbf{N}(t).\gamma(t) = 0$ but the latter cannot happen because $cN(t).\gamma(t) = -1$.