Hints / Solutions to Exercise sheet 1

Curves and Surfaces, MTH201

Question 1: Find a parametrization $\gamma(t)$ for a line segment joining two given points (x_1, y_1) and (x_2, y_2). Find $\dot{\gamma}(t)$.

Solution 1: The line points in the direction, $\mathbf{v} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$. Every other point on the line is a translate of (x_1, y_2) by scalar multiples of $\mathbf{v}$, i.e. $\gamma(t) = (x_1, y_1) + t\mathbf{v}$. When $t = 0$, we get the point (x_1, x_2) and when $t = 1$, we get the translate of (x_1, y_1) by $\mathbf{v}$, so we restrict the domain to $(0, 1)$ to get $\gamma : (0, 1) \to \mathbb{R}^2$.

Question 2: What does the parametrization trace out $\gamma(t) = (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$?

Solution 2: Observe that $(\frac{2t}{1+t^2})^2 + (\frac{1-t^2}{1+t^2})^2 = 1$, and, therefore, the image of the parametrization is a subset of $\{(x, y) \mid x^2 + y^2 = 1\}$, which is a circle. The parametrization is continuous so it will trace out an arc of the circle. The arc will be determined by the domain.

This shows that the circle can also be parametrized by a parametrization which uses only polynomials, more precisely "rational functions", i.e. functions that can be represented by quotients of polynomials.

Question 3: Show that the parametrization $\gamma(t) := (t^2 - 1, t(t^2 - 1))$ is not injective

Solution 3: $\gamma(1) = (0, 0) = \gamma(-1)$

Question 3: Express the curve traced out by the parametrization $\gamma(t) := (t^2 - 1, t(t^2 - 1))$ as the zero set of some function.

Solution 3: We will try to "eliminate the variable" t, which in this case is doable (but not always!)

$x = t^2 - 1$ and $y = t(t^2 - 1)$, so $y/x = t$ as long as $t \neq \pm 1$ (because we cannot divide by 0!). Now, plugging in $t = y/x$ in $x = t^2 - 1$, we get $x = (y/x)^2 - 1$, and so $x^3 = y^2 - x^2$. We know this holds for all t except ± 1, but for $t = \pm 1$ we get the point $(0, 0)$. We can easily check that this also satisfies $x^3 = y^2 - x^2$.

Question 4: For $\mathbf{v} : (\alpha, \beta) \to \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \to \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$.

Solution 4:

$\mathbf{v}(t) = (v_1(t), v_2(t))$

$\mathbf{w}(t) = (w_1(t), w_2(t))$

$\mathbf{v}(t).\mathbf{w}(t) = v_1(t)w_1(t) + v_2(t)w_2(t)$

$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1(t)w_1(t))' + (v_2(t)w_2(t))'$ (by definition of differentiating a function to $\mathbb{R}^2$)

So,
$(\mathbf{v}(t).\mathbf{w}(t))' = v_1'(t)w_1(t) + v_1(t)w_1'(t) + v_2'(t)w_2(t) + v_2(t)w_2(t)'$
Rearranging,
$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1'(t)w_1(t) + v_2'(t)w_2(t)) + (v_1(t)w_1'(t) + v_2(t)w_2(t)')$
$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1'(t), v_2'(t)).(w_1(t), w_2(t)) + (v_1(t), v_2(t)).(w_1'(t).w_2(t)')$
$(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

Question 5: If $\mathbf{n} : (\alpha, \beta) \to \mathbf{R}^2$ is such that $||\mathbf{n}(t)||$ is constant, then prove that $\dot{\mathbf{n}}(t)$ is either 0 or perpendicular to $\mathbf{n}(t)$.
Solution 5:
This question just generalizes what was seen $n(t).n(t) = C$
Differentiating,
$\dot{n}(t).n(t) + n(t).\dot{n}(t) = 0$
so, $2\dot{n}(t).n(t) = 0$
so, $\dot{n}(t).n(t) = 0$

Question 6: if we denote,

$$s_\alpha(t) := \int_{t_\alpha}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

$$s_\beta(t) := \int_{t_\beta}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

prove that $s_\beta(t) - s_\alpha(t)$ is a constant **(assume that $t_\alpha < t_\beta$).**
Solution 6:
This exercise is just saying that if you start measuring the distance traced out by your parametrization at time t_β rather than time t_α, you only need to add the distance covered from time t_α to time t_β We use the rule that,

$$\int_a^c f(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t + \int_b^c f(t)\mathrm{d}t$$

and therefore,

$$\int_a^c f(t)\mathrm{d}t - \int_b^c f(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t$$

So,

$$s_\alpha(t) := \int_{t_\alpha}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

$$s_\beta(t) := \int_{t_\beta}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

$$s_\beta(t) - s_\alpha(t) = \int_{t_\beta}^{t} ||\dot{\gamma}(u)||\mathrm{d}u - \int_{t_\alpha}^{t} ||\dot{\gamma}(u)||\mathrm{d}u = \int_{t_\alpha}^{t_\beta} ||\dot{\gamma}(u)||\mathrm{d}u$$

But the last integral is just a real number and does not depend on t so it is constant with respect to t.

Question 7: If $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth **and regular** parametrization, then show that $||\dot{\gamma}(t)|| : (\alpha, \beta) \to \mathbb{R}$ is smooth.
Solution 7:
We actually need to assume that γ is regular. Let $\gamma(t) = (x(t).y(t))$.
$\dot{\gamma}(t) = (\dot{x}(t).\dot{y}(t))$.
$\|\dot{\gamma}(t)\| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$.

$x(t)$ and $y(t)$ are smooth because that is the meaning of $\gamma(t)$ being smooth. Of course, even their derivatives are smooth, so $\dot{x}(t)$ and $\dot{y}(t)$ are smooth.

The squares of smooth functions are smooth, so $\dot{x}^2(t)$ and $\dot{y}^2(t)$ are smooth.

The sum of smooth functions is smooth, so $\dot{x}^2(t) + \dot{y}^2(t)$ is smooth.

We need to be careful about the square root function. Whenever $x > 0$, then if,

$$f(x) = \sqrt{x}$$

usin the rule for differentiating anything of the form x^n (in this case $x^{1/2}$),

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Note that at $x = 0$, this is undefined, and indeed it is not differentiable at $x = 0$. So we need to ensure that we are taking the square root of something which is strictly positive. But $\dot{x}^2(t) + \dot{y}^2(t) > 0$ except when $\dot{x}(t)$ and $\dot{y}(t)$ are *both* 0, in which case $\dot{\gamma}(t) = 0$ for that t, but that cannot happen with a regular parametrization.