Exercise sheet 6

Curves and Surfaces, MTH201

1. Consider any surface $S \subset \mathbb{R}^3$, let $f : S \to \mathbb{R}$, $f(x, y, z) = x$. Show that this is a smooth function.
2. Consider a surface $S \subset \mathbb{R}^3$, let $f : S \to P$, where $f(x, y, z) = (x, y, 0)$ and P is the plane defined by $z = 0$. Show that this is a smooth function.
3. Give a surface patch for a sphere. Compute the first fundamental form using the surface patch.
4. Give a surface patch for a plane. Compute the first fundamental form using the surface patch.
5. How does the matrix associated with the first fundamental form change under a coordinate transformation?
6. For the map f in exercise 2, and compute $\mathcal{D}_p(f)$.
7. This exercise is to revise what is taught in the lecture. Recall the definition of the Weingarten map, which is denoted by $\mathcal{W}_p$
   (a) Prove that $\mathcal{W}_p(\sigma_x) = \hat{\hat{\mathbf{n}}}_x$ and $\mathcal{W}_p(\sigma_y) = \hat{\hat{\mathbf{n}}}_y$
   (b) Prove that $\mathcal{W}(\mathbf{v})$ lies in the tangent space for any tangent vector v.
   (c) Prove that the Weingarten map is a linear map.
   (d) Therefore, it is enough to compute the Weingarten map for the basis tangent vectors σ_x and σ_y and we therefore need to find the coefficients a, b, c and d, below:
   $$\mathcal{W}_p(\sigma_x) = a\sigma_x + b\sigma_y$$
   $$\mathcal{W}_p(\sigma_y) = c\sigma_x + d\sigma_y$$
   (e) Define $L := \sigma_{xx}.\hat{\mathbf{n}}$, $M := \sigma_{xy}.\hat{\mathbf{n}}$, $N := \sigma_{yy}.\hat{\mathbf{n}}$. Prove that $L = -\sigma_x.\hat{\mathbf{n}}_x$, $M = -\sigma_x.\hat{\mathbf{n}}_y = -\sigma_y.\hat{\mathbf{n}}_x$, and $N = -\sigma_y.\hat{\mathbf{n}}_y$. The following matrix is called the matrix of the matrix of the second fundamental form:
   $$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$
   (f) By taking the dot product on both sides of each of the two equations in part 4 by σ_x and then by σ_y, obtain 4 linear equations whose unknowns are a, b, c, and d. Compute a, b, c, and d in terms of the first fundamenal form (i.e. E, F, G), and the second fundamental form (L, M, and N).