Exercise sheet 5

Curves and Surfaces, MTH201

Additional exercises

NOTE: These exercises repeat many of the concepts / exrecises covered earlier and are meant for you to identify gaps in your understanding. It includes concepts covered during the lectures by breaking them down into smaller exercises. They are not exhaustive and the mid-semester examination will not be restricted to these questions, however, hopefully these questions will help you to revise some of the concepts.

Curves on surfaces

Let $S \subset \mathbb{R}^3$ be a part of a surface and $\sigma : U \to S$ be a regular surface patch.

1. For each of the surface patches below, identify the surface that they (partially) cover:
   (a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, 0)$.
   (b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$.
   (c) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$.
   (d) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 - y^2})$.
   (e) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 + y^2})$.
2. If U is an open subset in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a smooth function, then show that $\sigma(x, y) := (x, y, f(x, y))$ is a regular surface patch.
3. Consider a $\gamma : (a, b) \to S \subset \mathbb{R}^3$ parametrizing a curve that lies on the part of the surface covered by the surface patch. In other words, for each t, $\gamma(t)$ must, be in the image of σ, i.e. there is some $x(t)$, and $y(t)$ in U, so that $\gamma(t) = \sigma(x(t), y(t))$. Assuming that $x(t)$ and $y(t)$ are smooth,
   (a) Consider the part of the surface covered by $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$ and consider the curve $\gamma(t) = (0, 0, t)$. Note that it lies on the surface. Write it in the form, $\gamma(t) = \sigma(x(t), y(t))$ by finding suitable functions $x(t)$ and $y(t)$. Do the same for the curve $\gamma_2(t) = (\cos(t), -\sin(t), 0)$ which also lies on the surface.
   (b) Show that

$$\dot{\gamma}(t_0) = x'(t_0)\sigma_x(x(t_0), y(t_0)) + y'(t_0)\sigma_y(x(t_0), y(t_0))$$

4. Show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are each velocity vectors of curves that lie on the surface. Why are they linearly independent?

5. Why do the previous two exercises show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are a basis for the tangent vectors?

6. Compute $\hat{\mathbf{n}}(p)$ for any point p on a plane. Show that it is constant.

7. Compute $\hat{\mathbf{n}}(p)$ for any point p on a sphere.

8. Consider a point p on the part of the surface covered by a surface patch. Therefore, it is of the form $p = \sigma(x_0, y_0)$ for some x_0 an y_0. Consider

$$\hat{\mathbf{n}}(p) = \frac{\sigma_x(x_0, y_0) \times \sigma_y(x_0, y_0)}{\|\sigma_x(x_0, y_0) \times \sigma_y(x_0, y_0)\|}$$

which is a vector in $\mathbb{R}^3$ based at p.

   (a) Is it a tangent vector? Why or why not?

   (b) Why is its dot product with $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ zero?

   (c) Why is its dot product with *any* tangent vector (of the surface at p) zero?

9. Consider a smooth function from the surface to $\mathbb{R}$, $f : S \to \mathbb{R}$. Show that the rate of change along any parametrization γ, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} f(\gamma(t)))$ depends on the partial derivatives of f at the point $\gamma(t_0)$ and the velocity of γ at t_0. (*Hint:* This is just a way of interpreting chain rule)

10. Consider $\hat{\tilde{\mathbf{n}}}(x, y) = \hat{\mathbf{n}}(\sigma(x, y))$. Note that if $p = \sigma(x, y)$, then $\hat{\tilde{\mathbf{n}}}(x, y) = \hat{\mathbf{n}}(p)$, i.e. $\hat{\tilde{\mathbf{n}}}$ is simply $\hat{\mathbf{n}}$ written in terms of the coordinates provided by σ. Note also that if $\gamma(t) = \sigma(x(t), y(t))$, then $\hat{\tilde{\mathbf{n}}}(x(t), y(t)) = \hat{\mathbf{n}}(\gamma(t))$.

    (a) Show that the rate of change of $\hat{\mathbf{n}}$ along a parametrization γ of a curve on the surface (i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \hat{\mathbf{n}}(\gamma(t))))$ depends only on $\hat{\mathbf{n}}$ at the point $\gamma(t_0)$ and the velocity of γ at t_0. (*Hint:* Apply the previous exercise to each coordinate of the function $\hat{\tilde{\mathbf{n}}}$. Why is it important to use $\hat{\tilde{\mathbf{n}}}$ and not $\hat{\mathbf{n}}$?)

11. From now on, we will assume that γ is a unit speed parametrization. Show that $\ddot{\gamma}(t_0).\hat{\mathbf{n}}(\gamma(t_0)) = -\dot{\gamma}(t_0).\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \hat{\mathbf{n}}(\gamma(t))$. Along with the previous exercise, this shows that the component of the acceleration in the direction of the normal, (denoted $\kappa_n(t_0)$) depends only on the normal to the surface and the direction of a unit speed parametrization.

12. Consider the other component of the acceleration, $\ddot{\gamma}(t_0) - \ddot{\gamma}(t_0).\hat{\mathbf{n}}(\gamma(t_0))$. Why is $\ddot{\gamma}(t_0)$ perpendicular to $\hat{\mathbf{n}}(\gamma(t_0))$? Why is $\ddot{\gamma}(t_0)$ parallel to $\mathbf{T}(t_0) \times \hat{\mathbf{n}}(\gamma(t_0))$? Let the magnitude be denoted by $\kappa_g(t_0)$

13. Show that $\kappa^2(t_0) = \kappa_n^2(t_0) + \kappa_g^2(t_0)$. (*Hint:* $\mathbf{T}(t_0)$, $\hat{\mathbf{n}}(\gamma(t_0))$, and $\mathbf{T}(t_0) \times \hat{\mathbf{n}}(\gamma(t_0))$ form an orthonormal basis in $\mathbb{R}^3$. Use the previous exercise.)

Applications of the Inverse Function Theorem

NOTE: This topic is a bit more advanced, however, luckily, you should be able to follow the rest of the course even if you do not understand the concepts covered in this section of the exercise set. In this course, the inverse function theorem is used only to justify smoothness of certain functions so that we are permitted to differentiate as much as we like. While it is certainly in the syllabus, since it is a an isolated and more difficult topic, it is better to worry about it only if you have understood the other more basic and more widely used concepts that are covered in the previous section of this exercise set.

1. Let $\sigma_1 : U_1 \to S$ and $\sigma_2 : U_2 \to S$ denote two regular surface patches that cover exactly the same part of the surface.

   (a) Let $\pi_{xy}(x, y, z) = (x, y)$, $\pi_{yz}(y, z) = (x, y)$, and $\pi_{xz}(x, z) = (x, y)$ denote the three projections obtained by dropping one of the coordinates. Show that the regularity of σ_1 at some point p implies that the Jacobian of $F_1(x, y) := \pi(\sigma_1(x, y))$ is invertible, where π denotes one of these three projections.

   (b) Show that the projection, π, from the previous exercise is injective when restricted to small enough neighbourhood of p.

   (c) Let $F_2(x, y) := \pi(\sigma_2(x, y))$. Show that $\Phi(x, y) = \sigma_1^{-1}(\sigma_2(x, y)) = F_1^{-1}(F_2(x, y))$.

   (d) Why is F_1^{-1} smooth? Therefore, why is Φ smooth?

2. We have already discussed that for a parametrization, γ, of a curve on a surface, $\gamma(t) = \sigma(x(t), y(t))$. Show that $x(t)$ and $y(t)$ are smooth functions. (*Hint:* Observe that $(x(t), y(t)) = \sigma^{-1}(\gamma(t))$. Use the "right" projection just as in the previous exercise to express this composition as a more convenient composition.)