Exercise sheet 8

Curves and Surfaces, MTH201

1. Let $f : S_1 \to S_2$ denote a smooth function that between surfaces that is 1-1, onto, its inverse is smooth, and so that $f^*\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ (called a local isometry) and let $\sigma_1 : U \to S_1$ denote a surface patch on S_1. Let $\sigma_2 = f \circ \sigma_1$
   (a) Show that $(\sigma_2)_x = D_p(f)(\sigma_1)_x$ and $(\sigma_2)_y = D_p(f)(\sigma_1)_y$
   (b) Show that if $(\sigma_1)_x \times (\sigma_1)_y \neq 0$, then $(\sigma_2)_x \times (\sigma_2)_y \neq 0$
   (c) We can then treat σ_2 as a surface patch for S_2. Show that if E_1, F_1, G_1 denote the entries of the matrix of the first fundamental form with respect to σ_1 and E_2, F_2, G_2 denote the entries of the matrix of the first fundamental form with respect to σ_2, then $E_1 = E_2$, $F_1 = F_2$, and $G_1 = G_3$.
   (d) Why does f map geodesics to geodesics?

2. Prove that the geodesic curvature of a curve in a plane (treated as a surface in $\mathbb{R}^3$) is equal to the plane curvature.

3. Compute the normal curvature of any curve on the sphere at a point in the region covered by the surface patch, $\sigma(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. Can you interpret the answer physically? Using this, prove that curves on the sphere that have constant geodesic curvature are circles.