Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,

We will define smooth functions on surfaces

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,

We study the surface using a patch

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,

Which contains p

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$

We view the surface in terms of a patch

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$

Now its domain is a subset of $\mathbb{R}^2$

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

so we know what it means to be smooth

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that,

Of course, we need to check that it does not depend on the chosen patch

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$,
where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse
is smooth,

This will always happen but we will prove it later

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth,

Let us examine the relationship between $f \circ \sigma$ and $f \circ \tilde{\sigma}$

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$

We know the relationship between the two patches

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Composing with f

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore,

Since the composition of smooth functions is smooth

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p
if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth

$\blacksquare$

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies f \circ \tilde{\sigma}$ is smooth

■

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

∎

Definition.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function**

We now similarly study functions between surfaces via their surface patches

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

■

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$

This time there is a surface patch not just for the domain

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

■

.

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,

but also for the co-domain

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,

This time we also compose by σ_2^{-1} so that the the input and output are from U_1 and U_2, respectively

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

■

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches
$\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition.

This exercise tells us why the definition does not depend on the choice of patches

.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$

f naturally defines a map on the tangent spaces as we shall now see

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

∎

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$
so that $f(p) = q$ for some $p \in S_1$ and $q \in S_2$.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

■

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$
so that $f(p) = q$ for some $p \in S_1$ and $q \in S_2$.
Let $\mathbf{v} \in T_p(S_1)$ denote a tangent vector at p.

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$
so that $f(p) = q$ for some $p \in S_1$ and $q \in S_2$.
Let $\mathbf{v} \in T_p(S_1)$ denote a tangent vector at p.
i.e. $\mathbf{v} = \dot{\gamma}(t_0)$ for some $\gamma : (\alpha, \beta) \to S_1$ and $t_0 \in (\alpha, \beta)$.

As usual, the tangent vector is a velocity vector of some curve on the surface

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$
so that $f(p) = q$ for some $p \in S_1$ and $q \in S_2$.
Let $\mathbf{v} \in T_p(S_1)$ denote a tangent vector at p.
i.e. $\mathbf{v} = \dot{\gamma}(t_0)$ for some $\gamma : (\alpha, \beta) \to S_1$ and $t_0 \in (\alpha, \beta)$.
Define $\mathrm{d}_p f : T_p(S_1) \to T_p(S_2)$
(where $T_p(S)$ denotes the tangent space of S at p)
by $\mathrm{d}_p f(\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) \in T_p(S_2)$

We simply consider the velocity vector of the image of that curve

Smooth functions

Definition. $f : S \to \mathbb{R}$ is called a **smooth map** at p if,
given a (regular) surface patch $\sigma : U \to S$,
so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$,
$f \circ \sigma$ is smooth at (x_0, y_0).

If $\tilde{\sigma} : \tilde{U} \to S$ is another surface patch so that, $\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth, invertible, and the inverse is smooth, Since,

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$f \circ \tilde{\sigma} = f \circ \sigma \circ \Phi$$

Therefore, $f \circ \sigma$ smooth $\implies$ $f \circ \tilde{\sigma}$ is smooth (because, $f \circ \sigma$ and Φ are smooth)

Definition. $f : S_1 \to S_2$ is said to be a **smooth function** at $p \in S_1$ if, given (regular) surface patches $\sigma_1 : U \to S_1$
(so that $p \in \sigma(U)$, $p = \sigma(x_0, y_0)$)
and $\sigma_2 : U \to S_2$,
$\sigma_2^{-1} \circ f \circ \sigma_1$ is smooth.

Exercise. Show that the definion of a smooth map does not depend on the choice of parametrizations.

Definition. Consider a smooth map, $f : S_1 \to S_2$
so that $f(p) = q$ for some $p \in S_1$ and $q \in S_2$.
Let $\mathbf{v} \in T_p(S_1)$ denote a tangent vector at p.
i.e. $\mathbf{v} = \dot{\gamma}(t_0)$ for some $\gamma : (\alpha, \beta) \to S_1$ and $t_0 \in (\alpha, \beta)$.
Define $\mathrm{d}_p f : T_p(S_1) \to T_p(S_2)$
(where $T_p(S)$ denotes the tangent space of S at p)
by $\mathrm{d}_p f(\mathbf{v}) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) \in T_p(S_2)$

and define that to be the image of $\mathbf{v}$ under $\mathrm{d}_p f$

$f : S_1 \to S_2,$

We now try to describe $d_p f$ in terms of the surface patch

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x, y))) = (g_1(x, y), g_2(x, y))$$

Here is f in terms of the surface patch

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x,y))) = (g_1(x,y), g_2(x,y))$$

$$f(\sigma_1(x(t),y(t))) = \sigma_2(g_1(x(t),y(t)), g_2(x(t),y(t)))$$

And this is the image of γ under f

.

$$f : S_1 \to S_2,$$

$$\sigma_2^{-1}(f(\sigma_1(x,y))) = (g_1(x,y), g_2(x,y))$$

$$f(\sigma_1(x(t), y(t))) = \sigma_2(g_1(x(t), y(t)), g_2(x(t), y(t)))$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\sigma_1(x(t), y(t))) = g_1'(x(t), y(t))\sigma_{2x} + g_2'(x(t), y(t))\sigma_{2y}$$

Written in a form that will allow us to write it in terms of σ_{2x} and σ_{2y}

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x,y))) = (g_1(x,y), g_2(x,y))$$

$$f(\sigma_1(x(t),y(t))) = \sigma_2(g_1(x(t),y(t)), g_2(x(t),y(t)))$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(\sigma_1(x(t),y(t))) &= g_1'(x(t),y(t))\sigma_{2x} + g_2'(x(t),y(t))\sigma_{2y} \\ &= (x'(t)g_{1x}(x(t),y(t)) + y'(t)g_{1y}(x(t),y(t))\sigma_x(x(t),y(t)) \\ &+ (x'(t)g_{2x}(x(t),y(t)) + y'(t)g_{2y}(x(t),y(t))\sigma_y(x(t),y(t))\end{aligned}$$

Now we apply chain rule to each coefficient

.

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x,y))) = (g_1(x,y), g_2(x,y))$$

$$f(\sigma_1(x(t),y(t))) = \sigma_2(g_1(x(t),y(t)), g_2(x(t),y(t)))$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(\sigma_1(x(t),y(t))) &= g_1'(x(t),y(t))\sigma_{2x} + g_2'(x(t),y(t))\sigma_{2y} \\ &= (x'(t)g_{1x}(x(t),y(t)) + y'(t)g_{1y}(x(t),y(t))\sigma_x(x(t),y(t)) \\ &+ (x'(t)g_{2x}(x(t),y(t)) + y'(t)g_{2y}(x(t),y(t))\sigma_y(x(t),y(t))\end{aligned}$$

In terms of coordinates,

$$= \begin{pmatrix} g_{1x}(t) & g_{1y}(t) \\ g_{2x}(t) & g_{2y}(t) \end{pmatrix}$$

And write it in terms of coordinates

.

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x, y))) = (g_1(x, y), g_2(x, y))$$

$$f(\sigma_1(x(t), y(t))) = \sigma_2(g_1(x(t), y(t)), g_2(x(t), y(t)))$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} f(\sigma_1(x(t), y(t))) &= g_1'(x(t), y(t))\sigma_{2x} + g_2'(x(t), y(t))\sigma_{2y} \\ &= (x'(t)g_{1x}(x(t), y(t)) + y'(t)g_{1y}(x(t), y(t))\sigma_x(x(t), y(t)) \\ &+ (x'(t)g_{2x}(x(t), y(t)) + y'(t)g_{2y}(x(t), y(t))\sigma_y(x(t), y(t))\end{aligned}$$

In terms of coordinates,

$$= \begin{pmatrix} g_{1x}(t) & g_{1y}(t) \\ g_{2x}(t) & g_{2y}(t) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

■

.

$f : S_1 \to S_2,$

$$\sigma_2^{-1}(f(\sigma_1(x,y))) = (g_1(x,y), g_2(x,y))$$

$$f(\sigma_1(x(t),y(t))) = \sigma_2(g_1(x(t),y(t)), g_2(x(t),y(t)))$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(\sigma_1(x(t),y(t))) &= g_1'(x(t),y(t))\sigma_{2x} + g_2'(x(t),y(t))\sigma_{2y} \\ &= (x'(t)g_{1x}(x(t),y(t)) + y'(t)g_{1y}(x(t),y(t))\sigma_x(x(t),y(t)) \\ &+ (x'(t)g_{2x}(x(t),y(t)) + y'(t)g_{2y}(x(t),y(t))\sigma_y(x(t),y(t))\end{aligned}$$

In terms of coordinates,

$$\begin{aligned}&= \begin{pmatrix} g_{1x}(t) & g_{1y}(t) \\ g_{2x}(t) & g_{2y}(t) \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\ &= J(\sigma_2^{-1} \circ f \circ \sigma_1) \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}\end{aligned}$$

Notice that the familiar Jacobian matrix shows up again

.

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,

The inner product of two tangent vectors is simply the dot product

.

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$\blacksquare$

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

The angular bracket notation only emphasizes that $\mathbf{v}_1$ and $\mathbf{v}_2$ must be tangent vectors

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

We will try to express this in terms of the surface patch

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0)$$

First note that by definition they are velocity vectors

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0)$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0)$$

$\blacksquare$

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0)$$

And now we use chain rule to express them in terms of σ_x and σ_y

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1.\mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0))$$

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1.\mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0))$$

■

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

■

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbf{v}_1 . \mathbf{v}_2$$

Finally, using them in the dot product

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 . \mathbf{v}_2 \\ &= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))\end{aligned}$$

$\blacksquare$

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1.\mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1.\mathbf{v}_2 \\ &= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\ &= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\ &+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0))\end{aligned}$$

Distributing and recognizing the appearance of E, F, and G

.

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 . \mathbf{v}_2 \\ &= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\ &= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\ &+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0))\end{aligned}$$

Observe that since $\mathbf{v}_1$ and $\mathbf{v}_2$ are based on the same point, $\gamma_1(t_0) = \gamma_2(t_0)$

.

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 . \mathbf{v}_2 \\
&= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\
&= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\
&+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0))
\end{aligned}$$

So $(x_1(t_0), y_1(t_0)) = (x_2(t_0), y_2(t_0))$

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1 . \mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1 . \mathbf{v}_2 \\ &= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\ &= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\ &+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0)) \\ &= \begin{pmatrix} x_1'(t_0) & y_1'(t_0) \end{pmatrix}\end{aligned}$$

But now observe that this can be expressed in matrix form

.

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1.\mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1.\mathbf{v}_2 \\ &= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\ &= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\ &+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0)) \\ &= \begin{pmatrix} x_1'(t_0) & y_1'(t_0) \end{pmatrix} \begin{pmatrix} E(x(t_0), y(t_0)) & F(x(t_0), y(t_0)) \\ F(x(t_0), y(t_0)) & G(x(t_0), y(t_0)) \end{pmatrix}\end{aligned}$$

∎

For $\mathbf{v}_1, \mathbf{v}_2 \in T_p(S)$,
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle := \mathbf{v}_1.\mathbf{v}_2$

$$\mathbf{v}_1 = \dot{\gamma}_1(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_1(t_0), y_1(t_0)) = x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))$$
$$\mathbf{v}_2 = \dot{\gamma}_2(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(x_2(t_0), y_2(t_0)) = x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0))$$

$$\begin{aligned}
\langle \mathbf{v}_1, \mathbf{v}_2 \rangle &= \mathbf{v}_1.\mathbf{v}_2 \\
&= (x_1'(t_0)\sigma_x(x_1(t_0), y_1(t_0)) + y_1'(t_0)\sigma_y(x_1(t_0), y_1(t_0))).(x_2'(t_0)\sigma_x(x_2(t_0), y_2(t_0)) + y_2'(t_0)\sigma_y(x_2(t_0), y_2(t_0)) \\
&= x_1'(t_0)x_2'(t_0)E(x(t_0), y(t_0)) + x_1'(t_0)y_2'(t_0)F(x(t_0), y(t_0)) \\
&+ y_1'(t_0)x_2'(t_0)F(x(t_0), y(t_0)) + y_1'(t_0)y_2'(t_0)G(x(t_0), y(t_0)) \\
&= \begin{pmatrix} x_1'(t_0) & y_1'(t_0) \end{pmatrix} \begin{pmatrix} E(x(t_0), y(t_0)) & F(x(t_0), y(t_0)) \\ F(x(t_0), y(t_0)) & G(x(t_0), y(t_0)) \end{pmatrix} \begin{pmatrix} x_2'(t_0) \\ y_2'(t_0) \end{pmatrix}
\end{aligned}$$

■

.

Surface	Surface patch

We will summarize how various concepts appear in terms of surface patches

Surface	**Surface patch**
$p \in S$	

A surface patch gives two coordinates to every point on part of a surface

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$

$\blacksquare$

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	

To every subset in the patch of S, it associates a subset in U

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$

■

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	

It associates to every curve on that part of the surface, a curve in U

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$

$\blacksquare$

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	

It provides a basis σ_x and σ_y, and tangent vectors are written in terms of them

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$

■

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	

To a function with domain S_1, it associates a function with domain U

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$

■

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	

To a function with surfaces as both domains and ranges, it associates a function between the domains of their pat

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$

∎

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	

To the inner product, it associates the matrix of "first fundamental form"

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$

$\blacksquare$

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$

■

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x'_1 & y'_1 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix}$, where $\mathbf{v}_i = x'_i\sigma_x + y'_i\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	

To a derivative of a function between two surfaces, it associates the "Jacobian" matrix

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1_x} & g_{1_y} \\ g_{2_x} & g_{2_y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$

■

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$
$\lVert \sigma_x \times \sigma_y \rVert$	

To the infinitesimal area, it associates the determinant of the first fundamental form matrix

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1_x} & g_{1_y} \\ g_{2_x} & g_{2_y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$
$\Vert\sigma_x \times \sigma_y\Vert$	$\lvert EG - F^2 \rvert = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

$\blacksquare$

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$
$\lVert \sigma_x \times \sigma_y \rVert$	$\lvert EG - F^2 \rvert = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix}$
$\text{Area} = \int_{\sigma(U)} \lVert \sigma_x \times \sigma_y \rVert$	

And to the area, the integral of the above determinant.

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$
$\\|\sigma_x \times \sigma_y\\|$	$\|EG - F^2\| = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix}$
Area $= \int_{\sigma(U)} \\|\sigma_x \times \sigma_y\\|$	$\int_U \|EG - F^2\|$

∎

Surface	**Surface patch**
$p \in S$	$(x, y) \in U$, where $\sigma(x, y) = p$
$A \subset S$	$B \subset U$, where $\sigma(B) = A$
$\gamma : (\alpha, \beta) \to S$	$\delta : (\alpha, \beta) \to U$, where $\gamma = \sigma \circ \delta$
$\mathbf{v} = \dot{\gamma}(t_0)$	$\mathbf{v} = x'\sigma_x + y'\sigma_y$, where $\gamma(t) = \sigma(x(t), y(t))$
$f : S_1 \to \mathbb{R}$	$g : U \to \mathbb{R}$, where $g = f \circ \sigma$
$f : S_1 \to S_2$	$g : U_1 \to U_2$, where $g = \sigma_2^{-1} \circ f \circ \sigma_1$
$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$	$\begin{pmatrix} x_1' & y_1' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$, where $\mathbf{v}_i = x_i'\sigma_x + y_i'\sigma_y$
$\mathrm{d}_p(f) : T_p(S_1) \to T_{f(p)}(S_2)$	$\begin{pmatrix} g_{1x} & g_{1y} \\ g_{2x} & g_{2y} \end{pmatrix}$, where $(g_1, g_2) = \sigma_2^{-1} \circ f \circ \sigma_1$
$\lVert \sigma_x \times \sigma_y \rVert$	$\lvert EG - F^2 \rvert = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix}$
Area $= \int_{\sigma(U)} \lVert \sigma_x \times \sigma_y \rVert$	$\int_U \lvert EG - F^2 \rvert$

■