$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$

Consider two surface patches whose images overlap

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$

By shrinking the domains if necessary, we may assume that their images are equal

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$\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth

and we can define a coordinate transformation to relate the two

$\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

We denote f and g to be the coordinates of the coordinate transformation

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

Remember that the domains have been shrunk so that Φ maps $\tilde{U}$ onto U

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

Remember that each patch gives us specially defined basis vectors.

$\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

How do the basis given by one patch relate with the other?

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$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}} \sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}} \sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

This is exactly what chain rule tells us when we take the derivatives on both sides of $\tilde{\sigma} = \sigma \circ \Phi$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

Of course $\tilde{\sigma}_x$ is some linear combination of σ_x and σ_y since σ_x and σ_y form a basis

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

Chain rule tells us that the coefficients are $f_{\tilde{x}}$ and $g_{\tilde{x}}$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

The same holds for $\tilde{\sigma}_y$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$

We know that vectors are defined as velocity vectors of parametrizations of curves on surfaces

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) =$$

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

In terms of the basis σ_x and σ_y, chain rule tells us the coefficients

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$
$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$

What happens when we change the parametrization?

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$
$$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$$

$\blacksquare$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}} \sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}} \sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}} \sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}} \sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

How do the coefficients with respect to the new basis compare with the old?

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

Observe,
$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$

For each t, Φ sends the points in $\tilde{U}$ to U

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

Observe,
$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$

$$\dot{\gamma}(t) = \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t)))$$
$$+$$

Writing $\tilde{\sigma}_x$ in terms of the old basis

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$$
$$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$$

Observe,
$$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$$

$$\dot{\gamma}(t) = \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t)))$$
$$+ \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t)))$$

Same for $\tilde{\sigma}_y$

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\begin{aligned} \tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) &= f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y})) \\ \tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) &= f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y})) \end{aligned}$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

Observe,
$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$

$$\begin{aligned} \dot{\gamma}(t) &= \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\ &\quad + \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\ &= \tilde{x}'(t)f_{\tilde{x}}\sigma_x(x(t), y(t)) + \tilde{x}'(t)g_{\tilde{x}}\sigma_y(x(t), y(t)) \\ &\quad + \tilde{y}'(t)f_{\tilde{y}}\sigma_x(x(t), y(t)) + \tilde{y}'(t)g_{\tilde{y}}\sigma_y(x(t), y(t)) \end{aligned}$$

Distributing everything and noting the highlighted part

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

Observe,
$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$

$$\begin{aligned}
\dot{\gamma}(t) &= \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t)))) \\
&\quad + \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t)))) \\
&= \tilde{x}'(t)f_{\tilde{x}}\sigma_x(x(t), y(t)) + \tilde{x}'(t)g_{\tilde{x}}\sigma_y(x(t), y(t)) \\
&\quad + \tilde{y}'(t)f_{\tilde{y}}\sigma_x(x(t), y(t)) + \tilde{y}'(t)g_{\tilde{y}}\sigma_y(x(t), y(t)) \\
&= (\tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}})\sigma_x(x(t), y(t)) \\
&\quad + (\tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}})\sigma_y(x(t), y(t))
\end{aligned}$$

Collecting terms to write everything in terms of σ_x and σ_y

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$$
$$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$$

Observe,

$$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$$

$$\begin{aligned}
\dot{\gamma}(t) &= \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\
&\quad + \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\
&= \tilde{x}'(t)f_{\tilde{x}}\sigma_x(x(t), y(t)) + \tilde{x}'(t)g_{\tilde{x}}\sigma_y(x(t), y(t)) \\
&\quad + \tilde{y}'(t)f_{\tilde{y}}\sigma_x(x(t), y(t)) + \tilde{y}'(t)g_{\tilde{y}}\sigma_y(x(t), y(t)) \\
&= (\tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}})\sigma_x(x(t), y(t)) \\
&\quad + (\tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}})\sigma_y(x(t), y(t))
\end{aligned}$$

$$x'(t) = \tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}}$$
$$y'(t) = \tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}}$$

Comparing coefficients

$\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\tilde{\sigma} = \sigma \circ \Phi$, where $\Phi : \tilde{U} \to U$ is smooth
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$\gamma(t) = \sigma(x(t), y(t))$
$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$

$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$
$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$

Observe,
$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$

$$\begin{aligned}\dot{\gamma}(t) &= \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\ &\quad + \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\ &= \tilde{x}'(t)f_{\tilde{x}}\sigma_x(x(t), y(t)) + \tilde{x}'(t)g_{\tilde{x}}\sigma_y(x(t), y(t)) \\ &\quad + \tilde{y}'(t)f_{\tilde{y}}\sigma_x(x(t), y(t)) + \tilde{y}'(t)g_{\tilde{y}}\sigma_y(x(t), y(t)) \\ &= (\tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}})\sigma_x(x(t), y(t)) \\ &\quad + (\tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}})\sigma_y(x(t), y(t))\end{aligned}$$

$x'(t) = \tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}}$
$y'(t) = \tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}}$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} f_{\tilde{x}} & f_{\tilde{y}} \\ g_{\tilde{x}} & g_{\tilde{y}} \end{pmatrix} \begin{pmatrix} \tilde{x}'(t) \\ \tilde{y}'(t) \end{pmatrix}$$

Writing in matrix form

$$\sigma : U \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$$
$$\tilde{\sigma} = \sigma \circ \Phi, \text{ where } \Phi : \tilde{U} \to U \text{ is smooth}$$
$$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$$
$$\Phi(\tilde{U}) = U$$

$$\tilde{\sigma}_{\tilde{x}}(\tilde{x}, \tilde{y}) = f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$
$$\tilde{\sigma}_{\tilde{y}}(\tilde{x}, \tilde{y}) = f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}, \tilde{y})) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}, \tilde{y}))$$

$$\gamma(t) = \sigma(x(t), y(t))$$
$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\gamma(t) = \tilde{\sigma}(\tilde{x}(t), \tilde{y}(t))$$
$$\dot{\gamma}(t) = \tilde{x}'(t)\tilde{\sigma}_x(\tilde{x}(t), \tilde{y}(t)) + \tilde{y}'(t)\tilde{\sigma}_y(\tilde{x}(t), \tilde{y}(t))$$

Observe,
$$(x(t), y(t)) = \Phi(\tilde{x}(t), \tilde{y}(t))$$

$$\begin{aligned}
\dot{\gamma}(t) &= \tilde{x}'(t)(f_{\tilde{x}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{x}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\
&\quad + \tilde{y}'(t)(f_{\tilde{y}}\sigma_x(\Phi(\tilde{x}(t), \tilde{y}(t))) + g_{\tilde{y}}\sigma_y(\Phi(\tilde{x}(t), \tilde{y}(t))) \\
&= \tilde{x}'(t)f_{\tilde{x}}\sigma_x(x(t), y(t)) + \tilde{x}'(t)g_{\tilde{x}}\sigma_y(x(t), y(t)) \\
&\quad + \tilde{y}'(t)f_{\tilde{y}}\sigma_x(x(t), y(t)) + \tilde{y}'(t)g_{\tilde{y}}\sigma_y(x(t), y(t)) \\
&= (\tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}})\sigma_x(x(t), y(t)) \\
&\quad + (\tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}})\sigma_y(x(t), y(t))
\end{aligned}$$

$$x'(t) = \tilde{x}'(t)f_{\tilde{x}} + \tilde{y}'(t)f_{\tilde{y}}$$
$$y'(t) = \tilde{x}'(t)g_{\tilde{x}} + \tilde{y}'(t)g_{\tilde{y}}$$

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} f_{\tilde{x}} & f_{\tilde{y}} \\ g_{\tilde{x}} & g_{\tilde{y}} \end{pmatrix} \begin{pmatrix} \tilde{x}'(t) \\ \tilde{y}'(t) \end{pmatrix}$$

This matrix associated with a smooth map will appear many times

Recall:

$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$,

We will need the following simple fact in our definition of area

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition.

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.

As usual, we give our surface two coordinates by a surface patch

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.

■

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.

$$\|\sigma_x(x, y) \times \sigma_y(x, y)\|$$

This cross product is the "infinitesimal" area

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.
$R \subset U$, specifies a region $\sigma(R) \subset \sigma(U) \subset S$

$$\|\sigma_x(x, y) \times \sigma_y(x, y)\|$$

We use the surface patch to specify a region

.

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.
$R \subset U$, specifies a region $\sigma(R) \subset \sigma(U) \subset S$

$$A := \int_R \|\sigma_x(x,y) \times \sigma_y(x,y)\| \mathrm{d}x\mathrm{d}y$$

and integrate, of course

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.
$R \subset U$, specifies a region $\sigma(R) \subset \sigma(U) \subset S$

$$A(R) := \int_R \|\sigma_x(x, y) \times \sigma_y(x, y)\| \mathrm{d}x\mathrm{d}y$$

Remember that this is the area of only a region

.

Recall:
$\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$, then $\|\mathbf{v}_1 \times \mathbf{v}_2\|$ is the area of the parallelogram with $\mathbf{v}_1$ and $\mathbf{v}_2$ as sides.

Definition. $\sigma : U \to S \subset \mathbb{R}^3$ a surface patch.
$R \subset U$, specifies a region $\sigma(R) \subset \sigma(U) \subset S$

$$A_\sigma(R) := \int_R \|\sigma_x(x, y) \times \sigma_y(x, y)\| \mathrm{d}x\mathrm{d}y$$

But it also seems to depend on the surface patch

Proposition. *A coordinate transformation leaves the area unchanged.*

However, it does not really depend on the surface patch

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

Consider a multivariable function f

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

How does its integral change if we "change the variables" by Φ

.

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

Φ changes the variables from x, y to $\tilde{x}$, $\tilde{y}$

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

We will denote the coordinates of Φ by f and g

.

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

This is the "change of variable formula" for integration

.

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Note the change in the region we integrating over it

.

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

□

As usual, we have two surface patches related by a coordinate transformation, Φ

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

□

■

.

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

□

■

Proposition. *A coordinate transformation leaves the area unchanged.*

> *Recall:*
> $h : A \subset \mathbb{R}^2 \to \mathbb{R}$
> $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
> $\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
>
> $$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

$$\tilde{\sigma}(\tilde{x}, \tilde{y}) = \sigma(\Phi(\tilde{x}, \tilde{y}))$$

□

■

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$

□

To simplify notation, we will use composition

Proposition. *A coordinate transformation leaves the area unchanged.*

$$\tilde{\sigma} = \sigma \circ \Phi$$

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

□

But do make sure that you check all the details

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:
$h : A \subset \mathbb{R}^2 \to \mathbb{R}$
$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$

□

Applying the chain rule

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$

$$\tilde{\sigma}_{\tilde{y}} = f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)$$

□

And also for σ_y

Proposition. *A coordinate transformation leaves the area unchanged.*

> *Recall:*
>
> $h : A \subset \mathbb{R}^2 \to \mathbb{R}$
>
> $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
>
> $\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
>
> $$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$

$$\tilde{\sigma}_{\tilde{y}} = f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)$$

$$\tilde{\sigma}_{\tilde{x}} \times \tilde{\sigma}_{\tilde{y}} = (f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)) \times (f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi))$$

□

This allows us to find out how the cross product relates

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$

$$\tilde{\sigma}_{\tilde{y}} = f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)$$

$$\begin{aligned} \tilde{\sigma}_{\tilde{x}} \times \tilde{\sigma}_{\tilde{y}} &= (f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)) \\ &\quad \times (f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)) \\ &= (f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})((\sigma_x \circ \Phi) \times (\sigma_y \circ \Phi)) \end{aligned}$$

□

Distributing the coefficients

Proposition. *A coordinate transformation leaves the area unchanged.*

> *Recall:*
> $h : A \subset \mathbb{R}^2 \to \mathbb{R}$
> $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$
> $\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
>
> $$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$
$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$
$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$
$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$
$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$
$$\tilde{\sigma}_{\tilde{y}} = f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)$$

$$\begin{aligned}\tilde{\sigma}_{\tilde{x}} \times \tilde{\sigma}_{\tilde{y}} &= (f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)) \\ &\quad \times (f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)) \\ &= (f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})((\sigma_x \circ \Phi) \times (\sigma_y \circ \Phi))\end{aligned}$$

□

Here, by $\sigma_x \times \sigma_y$ we mean a function $(a, b) \to \sigma_x(a, b) \times \sigma_y(a, b)$

Proposition. *A coordinate transformation leaves the area unchanged.*

Recall:

$h : A \subset \mathbb{R}^2 \to \mathbb{R}$

$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$$\int_R h = \int_{\Phi(R)} (h \circ \Phi)(f_{\tilde{x}} g_{\tilde{y}} - f_{\tilde{y}} g_{\tilde{x}})$$

Proof. $\sigma : U \to S \subset \mathbb{R}^3$

$\tilde{\sigma} : \tilde{U} \to S \subset \mathbb{R}^3$

$\Phi(\tilde{x}, \tilde{y}) = (f(\tilde{x}, \tilde{y}), g(\tilde{x}, \tilde{y}))$

$\Phi(\tilde{U}) = U$

$$\tilde{\sigma} = \sigma \circ \Phi$$

$$\tilde{\sigma}_{\tilde{x}} = f_{\tilde{x}}(\sigma_x \circ \Phi) + g_{\tilde{x}}(\sigma_y \circ \Phi)$$

$$\tilde{\sigma}_{\tilde{y}} = f_{\tilde{y}}(\sigma_x \circ \Phi) + g_{\tilde{y}}(\sigma_y \circ \Phi)$$

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□

This is why, $(\sigma_x \circ \Phi) \times (\sigma_y \circ \Phi) = (\sigma_x \times \sigma_y) \circ \Phi$

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This follows from change of variable formula of integration

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And this completes the proof that a coordinate transformation does not change the area

$\square$

Exercise.

$$A_\sigma(R) = \int_R \sqrt{E(x,y)G(x,y) - F^2(x,y)}\mathrm{d}x\mathrm{d}y$$

The area can be expressed entirely in terms of the first fundamental form (i.e. E, F, and G)

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