

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

It will be important to be absolutely clear about this

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$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

So we can write any vector as linear combination of them

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$$\alpha_i = v \cdot \mathbf{e}_i$$

And the coefficients can be recovered by a dot product

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If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$v \cdot \mathbf{e}_1 = 0$$

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A vector perpendicular to two of the basis vectors will have to be in the direction of the third

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We continue our study of the consequences of a curve being on a surfaces

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Once again, we can think of a curve using the usual coordinates

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or using the coordinates provided by the surface patch.

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let us now study acceleration and curvature of a curve on a surface

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$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

If the parametrization is unit speed, then $\ddot{\gamma}(t)$ is certainly perpendicular to $\mathbf{T}(t)$.

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$\sigma : U \rightarrow S$ a regular surface patch.

As usual, we study the surface with a surface patch

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A regular surface has a natural normal.

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

Try to solve this exercise (Hint: $\mathbf{T}(t)$ is in the span of?)

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The following discussion is motivated by this intuitive observation:

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The curve lies on a surface because

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$$\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))$$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

some component of the acceleration keeps the curve on the surface

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

This is obviously normal to the surface

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$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

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The magnitude of this components is called the normal curvature

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$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = ??$$

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If we subtract this component, what direction does the rest point in?

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As we have seen before, we can find that if we find two orthonormal vectors

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and then take the cross product

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The acceleration is already perpendicular to the unit tangent

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So is the component in the direction of the normal

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

Therefore, so is the difference

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because they both lie in the plane perpendicular to \mathbf{T}

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$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = ??$$

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$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

Also, since we have removed the component of the normal, it must be perpendicular to it

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

$$\alpha_2 = v \cdot \mathbf{e}_2 = 0$$

$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = ??$$

So we know that it will be perpendicular to $\hat{\mathbf{n}}$ and \mathbf{T}

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

And is therefore some scalar multiple of $\hat{\mathbf{n}} \times \mathbf{T}$

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

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$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.



$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

Curvature of a curve on a surface

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Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

Taking the dot product with $\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)$ on both sides of the above equation

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\ddot{\gamma}(t) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t))$$

Rearranging

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

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$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \quad (\text{normal curvature})$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

Plugging in the formula for normal curvature as defined above

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

Taking dot product with itself (on both sides)

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

But the left hand side is the square of the usual curvature!

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

Remember that when we viewed the curve simply as a space curve,

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

$\mathbf{T}(t)$, $\mathbf{N}(t)$, and $B(t) := \mathbf{T}(t) \times \mathbf{N}(t)$ proved to be a useful basis

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

If v is perpendicular to \mathbf{e}_1 and \mathbf{e}_2

$$\alpha_1 = v \cdot \mathbf{e}_1 = 0$$

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$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

Now, viewing the curve as a curve on the surface

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

$\mathbf{T}(t)$, $\hat{\mathbf{n}}(t)$, and $\mathbf{T}(t) \times \hat{\mathbf{n}}(t)$ proves to be a useful basis

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

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$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

Exercise. Prove that $\mathbf{N}(t) = \hat{\mathbf{n}}(t)$ if and only if $\kappa_g(t) = 0$

However, \mathbf{N} and $\hat{\mathbf{n}}$ need not coincide, as this simple exercise shows!

Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$$

$$\alpha_i = v \cdot \mathbf{e}_i$$

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$$\alpha_2 = v \cdot \mathbf{e}_2 = 0$$

$$\text{So, } v = \alpha_3 \mathbf{e}_3$$

$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$ unit speed

$$\ddot{\gamma}(t) \cdot \mathbf{T}(t) = 0$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$$\kappa_n(t) := \ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t)) \text{ (normal curvature)}$$

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t) \cdot (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned} \ddot{\gamma}(t) &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t) \cdot \hat{\mathbf{n}}(\gamma(t))) \hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t) (\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t) \hat{\mathbf{n}}(\gamma(t)) \end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

$$\kappa^2(t) = \kappa_g^2(t) + \kappa_n^2(t)$$

Exercise. Prove that $\mathbf{N}(t) = \hat{\mathbf{n}}(t)$ if and only if $\kappa_g(t) = 0$

Definition. A parametrization γ of a curve on a surface is called a geodesic if $\kappa_g(t) = 0$ for all t .

Such curves are called geodesics

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S$$

We will study the arc length of a curve on a surface

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

Again, since the surface is in \mathbb{R}^3 the curve is also in \mathbb{R}^3

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

We will now write everything in terms of a surface patch

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

Remember that $x(t)$ and $y(t)$ are the coordinates provided by σ

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

As we have seen, chain rule expresses the velocity vector in terms of σ_x and σ_y

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$



First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

So we can do the same for the dot with itself, to know its norm

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

Checking this formula should be a straightforward exercise

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

Let us abstract out the terms that refer to only the patch

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

Calling them E , F , and G

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

Note that E , F , and G are functions with domain U (i.e. domain of σ)

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

Observe that E , F , and G do not depend on the curve

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

They may be computed for a surface patch

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

and used for any curve we may want to study on that patch

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

just like we computed σ_x, σ_y and used it for the velocity of any curve

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

So the norm can be also be written in terms of E , F , and G ,

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$



First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$s(t) = \int_{t_0}^t \|\dot{\gamma}(t)\| dt$$

and, therefore, the arc length

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$\begin{aligned}s(t) &= \int_{t_0}^t \|\dot{\gamma}(t)\| dt \\ &= \int_{t_0}^t \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} dt\end{aligned}$$

can also be expressed in terms of E , F , and G

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$\begin{aligned}s(t) &= \int_{t_0}^t \|\dot{\gamma}(t)\| dt \\ &= \int_{t_0}^t \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} dt\end{aligned}$$

To summarize, we compute E , F , and G for each point of the surface patch and keep it aside

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$\begin{aligned}s(t) &= \int_{t_0}^t \|\dot{\gamma}(t)\| dt \\ &= \int_{t_0}^t \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} dt\end{aligned}$$

Given a curve, we take express it in terms of the surface patch

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

$$F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$$

$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$\begin{aligned}s(t) &= \int_{t_0}^t \|\dot{\gamma}(t)\| dt \\ &= \int_{t_0}^t \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} dt\end{aligned}$$

, i.e. find out its $(x(t), y(t))$

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

$\sigma : U \rightarrow S$ a regular surface patch.

$$\gamma(t) = \sigma(x(t), y(t))$$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)) \cdot \sigma_x(x(t), y(t))) \\ &\quad + 2x'(t)y'(t)(\sigma_x(x(t), y(t)) \cdot \sigma_y(x(t), y(t))) \\ &\quad + y'^2(t)(\sigma_y(x(t), y(t)) \cdot \sigma_y(x(t), y(t)))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t) \cdot \dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &\quad + 2x'(t)y'(t)F(x(t), y(t)) \\ &\quad + y'^2(t)G(x(t), y(t))\end{aligned}$$

$$\|\dot{\gamma}(t) \cdot \dot{\gamma}(t)\| = \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}$$

where,

$$E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$$

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$$G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$$

$$\begin{aligned}s(t) &= \int_{t_0}^t \|\dot{\gamma}(t)\| dt \\ &= \int_{t_0}^t \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} dt\end{aligned}$$

use that to find out its $x'(t)$ and $y'(t)$ and plug it into the above formula.

First fundamental form

$$\gamma : (\alpha, \beta) \rightarrow S \subset \mathbb{R}^3$$

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where,

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Observe,

$$\dot{\gamma}(t) \cdot \dot{\gamma}(t) = \begin{pmatrix} x'(t) & y'(t) \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Later, it will prove useful to know that E , F , and G can be arranged in a matrix