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Theorem. *Given,*

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1. *functions* $\tilde{\kappa} : (\alpha, \beta) \to \mathbb{R}$, $\tilde{\kappa}(t) > 0$ *for all* t, *and* $\tilde{\tau} : (\alpha, \beta) \to \mathbb{R}$
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products...

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