$\gamma:(\alpha,\beta)\to\mathbb{R}^2$

$$\begin{split} \gamma &: (\alpha, \beta) \to \mathbb{R}^2 \\ f(t) &= ||\dot{\gamma}(t)|| \end{split}$$

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The arc-length,

The arc-length, s from t = a to t = b

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The arc-length, s from t = a to t = b is approximated by,
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\begin{split} \gamma &: (\alpha, \beta) \to \mathbb{R}^2 \\ f(t) &= ||\dot{\gamma}(t)|| \\ f &: [\alpha, \beta] \to \mathbb{R} \text{ continuous} \end{split}
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The arc-length, s from t = a to t = b is approximated by, $f(t_1)(t_2 - t_1) +$

The arc-length, s from t = a to t = b is approximated by, $f(t_1)(t_2 - t_1) + f(t_2)(t_3 - t_2) +$

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The arc-length, s from t = a to t = b is approximated by, $||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \ldots + ||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})$

The arc-length, s from t = a to t = b is approximated by, $||\dot{\gamma}(t_1)||(t_2 - \underbrace{t_1}_{a}) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots +$ $||\dot{\gamma}(t_{n-1})||(\underbrace{t_n}_{a} - t_{n-1})$

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Better and better approximations "converge". Denoted,

 $||\dot{\gamma}(t)||$

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 $||\dot{\gamma}(t)||\mathrm{d}t$

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$$\int ||\dot{\gamma}(t)|| \mathrm{d}t$$

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$$\int_{t=a} ||\dot{\gamma}(t)|| \mathrm{d}t$$

The arc-length, s from t = a to t = b is approximated by, $||\dot{\gamma}(t_1)||(t_2 - \underbrace{t_1}_{a}) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots +$ $||\dot{\gamma}(t_{n-1})||(\underbrace{t_n}_{a} - t_{n-1})$

$$\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$$

$$\begin{split} \gamma : (\alpha, \beta) \to \mathbb{R}^2 \\ f(t) &= ||\dot{\gamma}(t)|| \\ f : [\alpha, \beta] \to \mathbb{R} \text{ continuous} \\ \end{split}$$
The arc-length, s from t = a to t = b is approximated by, $||\dot{\gamma}(t_1)||(t_2 - \underbrace{t_1}_{a}) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \ldots + ||\dot{\gamma}(t_{n-1})||(\underbrace{t_n}_{a} - t_{n-1})$

Better and better approximations "converge". Denoted,

$$\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$$

Arc length

Definition.

 $\gamma:(\alpha,\beta)\to\mathbb{R}^2$

Arc length

Definition.

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The arc-length, s from t = a to t = b is approximated

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$$||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + ||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})$$

$$\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$$

Arc length

Definition.

 $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular (of course!)

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The arc-length, s from t = a to t = b is approximated by.

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 $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular (of course!) parametrization.

Arc-length beginning at t_0 ,

The arc-length, s from t = a to t = b is approximated by,

$$\frac{||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + \frac{1}{a}}{||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})}$$

$$\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$$

The arc-length, s from t = a to t = b is approximated by

$$||\dot{\gamma}(t_1)||(t_2 - \underbrace{t_1}_{a}) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + \\ ||\dot{\gamma}(t_{n-1})||(\underbrace{t_n}_{a} - t_{n-1})$$

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Arc-length beginning at t_0 , denoted s(t),

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$$\begin{aligned} ||\dot{\gamma}(t_1)||(t_2 - \underbrace{t_1}_{a}) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + \\ ||\dot{\gamma}(t_{n-1})||(\underbrace{t_n}_{a} - t_{n-1}) \end{aligned}$$

Arc length

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Arc-length beginning at t_0 , denoted s(t),

$$s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d} u$$

Better and better approximations "converge". Denoted,

 $\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$

The arc-length, s from t = a to t = b is approximated by,

$$\frac{||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + ||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})||(t_n - t_{n-1})||$$

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 $\int_{t=a}$

The arc-length, s from t = a to t = b is approximated by,

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Arc length

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The arc-length, s from t = a to t = b is approximated by,

$$\frac{||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + u_n)}{||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})}$$

Arc length

Definition.

 $s_{\beta}(t)$

 $\gamma:(\alpha,\beta)\to\mathbb{R}^2$ is a smooth and regular (of course!) parametrization.

Arc-length beginning at t_0 , denoted s(t),

$$s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d} u$$

$$\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$$

$$s_{\alpha}(t) := \int_{t_{\alpha}}^{t} ||\dot{\gamma}(u)|| \mathrm{d}u$$

The arc-length, s from t = a to t = b is approximated by,

$$\frac{||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + u_n||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})||(t_n - t_{n-1})||$$

Arc length

Definition.

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Better and better approximations "converge". Denoted, Exercise.

 $\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$

$$s_{\alpha}(t) := \int_{t_{\alpha}}^{t} ||\dot{\gamma}(u)|| \mathrm{d}u$$

$$s_{\beta}(t) := \int_{t_{\beta}}^{t} ||\dot{\gamma}(u)|| \mathrm{d}u$$

The arc-length, s from t = a to t = b is approximated by,

$$\frac{||\dot{\gamma}(t_1)||(t_2 - t_1) + ||\dot{\gamma}(t_2)||(t_3 - t_2) + \dots + u_n)}{||\dot{\gamma}(t_{n-1})||(t_n - t_{n-1})}$$

Arc length

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Better and better approximations "converge". Denoted, Exercise.

 $\int_{t=a}^{t=b} ||\dot{\gamma}(t)|| \mathrm{d}t$

$$s_{\alpha}(t) := \int_{t_{\alpha}}^{t} ||\dot{\gamma}(u)|| \mathrm{d}u$$

$$s_{\beta}(t) := \int_{t_{\beta}}^{t} ||\dot{\gamma}(u)|| \mathrm{d}u$$

Prove that $s_{\beta}(t) - s_{\alpha}(t)$ is a constant.

Theorem (First Fundamental theorem of calculus).

f

Theorem (First Fundamental theorem of calculus). $f: [\alpha, \beta] \to \mathbb{R}$ continuous **Theorem** (First Fundamental theorem of calculus). $f: [\alpha, \beta] \to \mathbb{R}$ continuous

F(t)
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then, F'(t) = f(t)

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Corollary. $\gamma:(\alpha,\beta)\to\mathbb{R}^2$

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

and s(t) its arc-length function beginning at t_0 then,

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

Proof.

s(t)

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Proof.

$$s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$$

by the First Fundamental Theorem of Calculus,

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

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Corollary. The arc length function

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and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

Proof.

$$s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

Corollary. The arc length function s(t)

$$F(t):=\int_{t_0}^t f(u)\mathrm{d} u$$

then, F'(t) = f(t)

Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

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$$s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

Corollary. The arc length function s(t) is smooth.

Proof.

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

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Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth

$$F(t) := \int_{t_0}^t f(u) \mathrm{d} u$$

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

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Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? \Box

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

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Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular parametrization.

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Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If g(f(t)) = t

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and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

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Observe, If g(f(t)) = t, then

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

then, F'(t) = f(t)

Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If g(f(t)) = t, then q'(f(t)) f'(t) = 1

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Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If q(f(t)) = t, then **Corollary.** $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, therefore,

parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, s'(t) = $||\dot{\gamma}(t)||$

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

then, F'(t) = f(t)

Corollary. The arc length function s(t) is smooth.

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Observe, If q(f(t)) = t, then **Corollary.** $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, therefore, if $f'(t) \neq 0$ parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, s'(t) = $||\dot{\gamma}(t)||$

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

then, F'(t) = f(t)

Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If g(f(t)) = t, then g'(f(t))f'(t) = 1, therefore, if $f'(t) \neq 0$ for any

Corollary. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, therefore, if $f'(t) \neq 0$ for any t, parametrization.

and s(t) its arc-length function beginning at t_0 then, $s'(t) = ||\dot{\gamma}(t)||$

$$g'(f(t)) = \frac{1}{f'(t)}$$

Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

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Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, $s'(t) = ||\dot{\gamma}(t)||$

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

then, F'(t) = f(t)

Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If q(f(t)) = t, then **Corollary.** $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, therefore, if $f'(t) \neq 0$ for any t, $g'(f(t)) = \frac{1}{f'(t)}$

parametrization. and s(t) its arc-length function beginning at t_0 then,

 $s'(t) = ||\dot{\gamma}(t)||$

Proof.

 $s(t) := \int_{t_0}^t ||\dot{\gamma}(u)|| \mathrm{d}u$

by the First Fundamental Theorem of Calculus, s'(t) = $\|\dot{\gamma}(t)\|$

$$F(t) := \int_{t_0}^t f(u) \mathrm{d}u$$

then, F'(t) = f(t)

Corollary. The arc length function s(t) is smooth.

Proof. $s'(t) = ||\dot{\gamma}(t)||$ which is itself smooth (Why? exercise!).

Observe, If q(f(t)) = t, then **Corollary.** $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth and regular g'(f(t))f'(t) = 1, therefore, if $f'(t) \neq 0$ for any t, $g'(f(t)) = \frac{1}{f'(t)}$ Taking f(t) = s(t)

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 γ :

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 $\gamma:(\alpha,\beta)\to$

$$\begin{split} (s^{-1})'(\tilde{t}) &= \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{||\gamma'(s^{-1}(\tilde{t}))||}\\ \gamma: (\alpha, \beta) \to \mathbb{R}^2 \end{split}$$

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 $\begin{array}{c} \gamma: (\overline{\alpha, \beta}) \to \mathbb{R}^2 \\ \tilde{\gamma}: \end{array}$

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$$\begin{split} &\gamma:(\alpha,\beta)\to\mathbb{R}^2\\ &\tilde{\gamma}:(\tilde{\alpha},\tilde{\beta})\to\mathbb{R}^2\\ &\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t})) \end{split}$$

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If $\phi(t) = s^{-1}(t)$,

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$$\begin{split} & \text{If } \phi(t) = s^{-1}(t), \\ & \tilde{\gamma}'(\tilde{t}) = \gamma'(s^{-1}(\tilde{t}))(s^{-1})'(\tilde{t}) = \gamma'(s^{-1}(\tilde{t})) \frac{1}{||\gamma'(s^{-1}(\tilde{t}))||} \end{split}$$

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[]]Theorem.

 $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ a regular smooth parametrization s(t), the arc-length from t_0 to t



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Theorem.

 $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ a regular smooth parametrization s(t), the arc-length from t_0 to t (where $t, t_0 \in (\alpha, \beta)$)

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$^{-1}$ Theorem.

 $\gamma: (\alpha, \beta) \to \mathbb{R}^2$ a regular smooth parametrization s(t), the arc-length from t_0 to t (where $t, t_0 \in (\alpha, \beta)$) $\tilde{\gamma}(t) = \gamma(s^{-1}(t))$, then

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 $\tilde{\gamma}$ is a unit speed re-parametrization.