$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$

$$
\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
$$

$$
f(t)=\|\dot{\gamma}(t)\|
$$

$f$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta]$

$$
\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
$$

$$
f(t)=\|\dot{\gamma}(t)\|
$$

$$
f:[\alpha, \beta] \rightarrow \mathbb{R}
$$

$$
\begin{aligned}
& \gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2} \\
& f(t)=\|\dot{\gamma}(t)\| \\
& f:[\alpha, \beta] \rightarrow \mathbb{R} \text { continuous }
\end{aligned}
$$

The arc-length,
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$f\left(t_{1}\right)\left(t_{2}-t_{1}\right)+$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$f\left(t_{1}\right)\left(t_{2}-t_{1}\right)+f\left(t_{2}\right)\left(t_{3}-t_{2}\right)+$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$f\left(t_{1}\right)\left(t_{2}-t_{1}\right)+f\left(t_{2}\right)\left(t_{3}-t_{2}\right)+\ldots$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|\left(t_{2}-t_{1}\right)+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|\left(t_{n}-\right.$ $\left.t_{n-1}\right)$

```
\(\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}\)
\(f(t)=\|\dot{\gamma}(t)\|\)
\(f:[\alpha, \beta] \rightarrow \mathbb{R}\) continuous
The arc-length, \(s\) from \(t=a\) to \(t=b\) is approximated by,
\(\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|\left(t_{n}-\right.\) \(\left.t_{n-1}\right)\)
```

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$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$


Better and better approximations "converge".

$$
\begin{aligned}
& \gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2} \\
& f(t)=\|\dot{\gamma}(t)\| \\
& f:[\alpha, \beta] \rightarrow \mathbb{R} \text { continuous } \\
& \text { The arc-length, } s \text { from } t=a \text { to } t=b \text { is approximated } \\
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

Better and better approximations "converge". Denoted,

$$
\|\dot{\gamma}(t)\|
$$

$$
\begin{aligned}
& \gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2} \\
& f(t)=\|\dot{\gamma}(t)\| \\
& f:[\alpha, \beta] \rightarrow \mathbb{R} \text { continuous } \\
& \text { The arc-length, } s \text { from } t=a \text { to } t=b \text { is approximated } \\
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

Better and better approximations "converge". Denoted,

$$
\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int_{t=a}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,

$$
\begin{aligned}
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$

## Arc length

## Definition.

The arc-length, $s$ from $t=a$ to $t=b$ is approximated $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated
by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Definition.

## Arc length

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated
by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Definition.

## Arc length

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!)
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
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The arc-length, $s$ from $t=a$ to $t=b$ is approximated by,

$$
\begin{aligned}
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
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The arc-length, $s$ from $t=a$ to $t=b$ is approximated
by,
$\left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+$ Arc-length beginning at $t_{0}$,
$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$
Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Arc length

## Definition.

 parametrization.Arc-length beginning at $t_{0}$,
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!)
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!)

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+\text { Arc-length beginning at } t_{0} \text {, denoted } s(t),
\end{aligned}
$$

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated
by,
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Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.
Arc-length beginning at $t_{0}$, denoted $s(t)$,

$$
s(t)
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
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$\left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.

Better and better approximations "converge". Denoted,

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.

Better and better approximations "converge". Denoted, Exercise.

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.
Arc-length beginning at $t_{0}$, denoted $s(t)$,

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

## Exercise.

$$
\int_{t=a}^{t=b}\|\dot{\gamma}(t)\| \mathrm{d} t
$$

$$
s_{\alpha}(t)
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

## Arc length

## Definition.

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.
Arc-length beginning at $t_{0}$, denoted $s(t)$,

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

## Exercise.

$$
\begin{aligned}
& s_{\alpha}(t):=\int_{t_{\alpha}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u \\
& s_{\beta}(t)
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
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$$

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Arc-length beginning at $t_{0}$, denoted $s(t)$,

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

## Exercise.

$$
\begin{aligned}
& s_{\alpha}(t):=\int_{t_{\alpha}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u \\
& s_{\beta}(t):=\int_{t_{\beta}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$f(t)=\|\dot{\gamma}(t)\|$
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous
The arc-length, $s$ from $t=a$ to $t=b$ is approximated

$$
\begin{aligned}
& \text { by, } \\
& \left\|\dot{\gamma}\left(t_{1}\right)\right\|(t_{2}-\underbrace{t_{1}}_{a})+\left\|\dot{\gamma}\left(t_{2}\right)\right\|\left(t_{3}-t_{2}\right)+\ldots+ \\
& \left\|\dot{\gamma}\left(t_{n-1}\right)\right\|(\underbrace{t_{n}}_{a}-t_{n-1})
\end{aligned}
$$

## Arc length

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$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular (of course!) parametrization.
Arc-length beginning at $t_{0}$, denoted $s(t)$,

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

## Exercise.

$$
\begin{aligned}
& s_{\alpha}(t):=\int_{t_{\alpha}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u \\
& s_{\beta}(t):=\int_{t_{\beta}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
\end{aligned}
$$

Prove that $s_{\beta}(t)-s_{\alpha}(t)$ is a constant.

## Theorem (First Fundamental theorem of calculus).

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$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t)
$$

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

then,

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

then, $F^{\prime}(t)=f(t)$

Theorem (First Fundamental theorem of calculus).
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

$$
\text { then, } F^{\prime}(t)=f(t)
$$

Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$

Theorem (First Fundamental theorem of calculus).
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

$$
\text { then, } F^{\prime}(t)=f(t)
$$

Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

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Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.

Theorem (First Fundamental theorem of calculus).
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
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then, $F^{\prime}(t)=f(t)$
Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.
and $s(t)$ its arc-length function beginning at $t_{0}$ then,

Theorem (First Fundamental theorem of calculus).
$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

then, $F^{\prime}(t)=f(t)$
Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.
and $s(t)$ its arc-length function beginning at $t_{0}$ then, $s^{\prime}(t)=\|\dot{\gamma}(t)\|$

## Proof.

$s(t)$

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
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Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.
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Proof.

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
$$

then, $F^{\prime}(t)=f(t)$
Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.
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Proof.

$$
s(t):=\int_{t_{0}}^{t}\|\dot{\gamma}(u)\| \mathrm{d} u
$$

by the First Fundamental Theorem of Calculus,

## Theorem (First Fundamental theorem of calculus).

$f:[\alpha, \beta] \rightarrow \mathbb{R}$ continuous

$$
F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
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Proof.

$$
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by the First Fundamental Theorem of Calculus, $s^{\prime}(t)=$ $\|\dot{\gamma}(t)\|$

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F(t):=\int_{t_{0}}^{t} f(u) \mathrm{d} u
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then, $F^{\prime}(t)=f(t)$
Corollary. $\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ is a smooth and regular parametrization.
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g^{\prime}(f(t))=\frac{1}{f^{\prime}(t)}
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$g^{\prime}(f(t))=\frac{1}{f^{\prime}(t)}$
Taking $f(t)=s(t)$ and $g(t)=s^{-1}(t)$

$$
\left(s^{-1}\right)^{\prime}(s(t))=\frac{1}{s^{\prime}(t)}
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$\gamma$ :

$$
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$$
\gamma:(\alpha, \beta) \rightarrow
$$

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$$
\tilde{\gamma}:
$$

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$$

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\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$$
\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}(\tilde{t})
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$$
\gamma:(\alpha, \underset{\sim}{\beta}) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
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$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$

$$
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\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
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\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}
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\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))
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$$
\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
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If $\phi(t)=s^{-1}(t)$,

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
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$$
\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
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$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t), \\
& \tilde{\gamma}^{\prime}(\tilde{t})
\end{aligned}
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
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$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
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$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
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$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$\left\|\tilde{\gamma}^{\prime}(\hat{t})\right\|$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem } .
\end{aligned}
$$

$$
\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}
$$

$$
\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))
$$

$$
\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
$$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem } .
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem } .
\end{aligned}
$$

$$
\begin{aligned}
& \gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2} \\
& \tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2} \\
& \tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t})) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
\end{aligned}
$$

$$
\text { If } \phi(t)=s^{-1}(t)
$$

$$
\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem }
\end{aligned}
$$

$\gamma:(\alpha, \beta \tilde{\tilde{\beta}}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ a regular smooth parametrization $s(t)$,

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem }
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ a regular smooth parametrization $s(t)$, the arc-length from $t_{0}$ to $t$

$$
\begin{aligned}
& \tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t})) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
\end{aligned}
$$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem }
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ a regular smooth parametrization $s(t)$, the arc-length from $t_{0}$ to $t$ (where $t, t_{0} \in(\alpha, \beta)$ )

$$
\begin{aligned}
& \tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t})) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
\end{aligned}
$$

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem }
\end{aligned}
$$

$$
\begin{aligned}
& \gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2} \\
& \tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2} \\
& \tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t})) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ a regular smooth parametrization $s(t)$, the arc-length from $t_{0}$ to $t$ (where $t, t_{0} \in(\alpha, \beta)$ ) $\tilde{\gamma}(t)=\gamma\left(s^{-1}(t)\right)$, then

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}
\end{aligned}
$$

$$
\left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
$$

$$
\left(s^{-1}\right)^{\prime}(\tilde{t})=\frac{1}{s^{\prime}\left(s^{-1}(\tilde{t})\right)}=\frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \quad \begin{aligned}
& \text { Proved that }, \\
& \text { Theorem } .
\end{aligned}
$$

$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}:(\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^{2}$
$\tilde{\gamma}(\tilde{t})=\gamma(\phi(\tilde{t}))$
$\tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}(\phi(\tilde{t})) \phi^{\prime}(\tilde{t})$
$\gamma:(\alpha, \beta) \rightarrow \mathbb{R}^{2}$ a regular smooth parametrization
$s(t)$, the arc-length from $t_{0}$ to $t$ (where $t, t_{0} \in(\alpha, \beta)$ )
$\tilde{\gamma}(t)=\gamma\left(s^{-1}(t)\right)$, then
$\tilde{\gamma}$ is a unit speed re-parametrization.

$$
\begin{aligned}
& \text { If } \phi(t)=s^{-1}(t) \\
& \tilde{\gamma}^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\left(s^{-1}\right)^{\prime}(\tilde{t})=\gamma^{\prime}\left(s^{-1}(\tilde{t})\right) \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|} \\
& \left\|\tilde{\gamma}^{\prime}(\tilde{t})\right\|=\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\| \frac{1}{\left\|\gamma^{\prime}\left(s^{-1}(\tilde{t})\right)\right\|}=1
\end{aligned}
$$

